Convolution, Impulse Response and Stability

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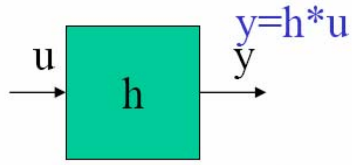
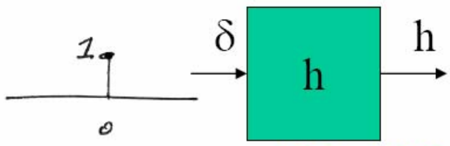
Recall Convolution

For each time k $$y[k] = \sum_{m=-\infty}^{\infty} u[m]h[k-m]$$

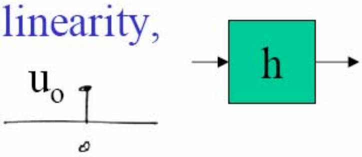


Consider the response to a unit pulse



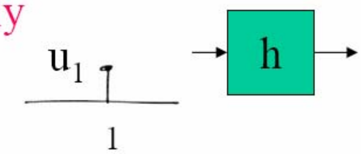
y[k]=h[k]

By linearity,



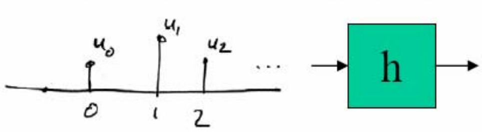
$y[k]=u_o h[k]$

Similarly



$y[k]=u_1 h[k-1]$

Then for the input signal ($u[k] = 0$ for $k < 0$)



$$y[k] = \sum_{m=0}^{\infty} u[m]h[k-m]$$

The input-output characteristics of an LTI system are completely described by the impulse response *h[n]*.

An LTI system with impulse response *h[n]* is invertible if there exists another impulse repsonse function $h_i[n]$ such that

$$h[n] * h_i[n] = \delta[n]$$

Ex. What is the inverse of $h[n] = 3\delta[n+5]$?

$$h_i[n] = \frac{1}{3}\delta[n-5]$$

Pulse Response

- The system is **causal** if h(k) is zero for all times k< 0 (output doesn't depend on future inputs—can see this from the convolution formula)
- If at least one value of h(k) for a negative k is not zero, the system is not causal

Stability and Pulse Response

- You can determine the stability of a causal system from its pulse response
  - BIBO stability: "bounded input, bounded output" –if you put finite signals in, you will get finite signals out

Stability and Pulse Response

By the convolution formula, for a bounded input *e[k]* we have

$$|e[k]| \le M < \infty \quad \textit{for all } k$$

$$|u[k]| \le \left|\sum_{m=-\infty}^{\infty} e[m]h[k-m]\right| \le \sum_{m=-\infty}^{\infty} |e[m]||h[k-m]| \le M \sum_{m=-\infty}^{\infty} |h[k-m]|$$

Thus the output will be bounded for every bounded input if

$$\sum_{m=-\infty}^{\infty} |h[k-m]| < \infty$$

or, equivalently

$$\sum_{m=-\infty}^{\infty} |h[k]| < \infty$$

Stability and Pulse Response

By the convolution formula, for a bounded input *e[k]* $|e[k]| \leq M < \infty \quad \textit{for all } k$
we have

$$|u[k]| \leq \left| \sum_{m=-\infty}^{\infty} e[m]h[k-m] \right| \leq \sum_{m=-\infty}^{\infty} |e[m]||h[k-m]| \leq M \sum_{m=-\infty}^{\infty} |h[k-m]|$$

Thus the output will be bounded for every bounded input if

$$\boxed{\sum_{m=-\infty}^{\infty} |h[k]| < \infty}$$

This is also a necessary condition—so, by looking at the pulse response you can determine causality and stability of the system

Ex. Is

$$h[n] = (\frac{1}{3})^n u[n]$$

BIBO stable?

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=0}^{\infty} (\tfrac{1}{3})^k = \frac{1}{1-\frac{1}{3}} < \infty$$

Yes

More examples: Causal? Stable?

1. $h_1[n] = u[n]$ (the accumulator)

$$\sum_{k=0}^{\infty} 1 = \infty$$

Not stable, but causal

2. $h_2[n] = 3^n u[n]$

Not stable, but causal

$$\sum_{k=0}^{\infty} 3^n = \infty$$

Not causal, but stable

3. $h_3[n] = (3)^n u[-n]$

$$\sum_{k=-\infty}^{0} 3^k = \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k = \frac{1}{1-\frac{1}{3}} < \infty$$

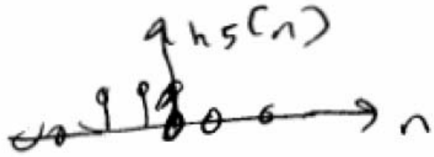
4. $h_4[n] = \cos(\frac{\pi}{3} n) u[n]$

$$\sum_{n=0}^{\infty} \left| \cos\left(\frac{\pi}{3} n\right) \right| = \infty$$

Causal, not stable

5. $h_5[n] = u[n+2] - u[n]$

Not Causal, but stable



Summary of DLTI Systems

System attributes:
Time-invariance, causality, linearity, invertibility
---Can be determined from impulse response

LTI system behavior specified by its impulse response
(convolution relating output to input and impulse response)

Can use superposition to break up input signal into parts
(and convolve separately, then recombine—use linearity to
make problems easier to solve

Can get step response from impulse response and vice versa.

Continuous Time	Discrete Time
$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$	$u[n] = \sum_{k=-\infty}^{n} \delta[k]$
$s(t) = \int_{-\infty}^{t} h(\tau)d\tau = h(t) * u(t)$	$s[n] = \sum_{k=-\infty}^{n} h[k] = h[n] * u[n]$
$\delta(t) = \frac{d}{dt}u(t)$	$\delta[n] = u[n] - u[n-1]$
$h(t) = \frac{d}{dt}s(t)$	$h[n] = s[n] - s[n-1]$