Block Diagram Analysis of Sampled Data Systems

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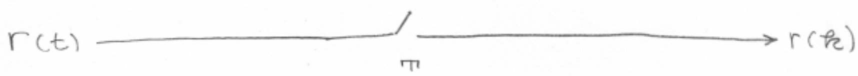
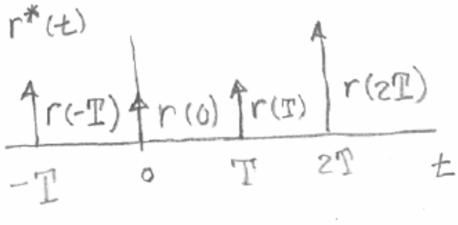
Laplace Transform of Sampled Signal





$$r^*(t) = r(t)\sum_{k=-\infty}^{\infty}\delta(t-kT)$$

The sampled signal *r*(t)* is an "impulse train" of continuous time impulses, one every *T*, height determined by corresponding *r* values at the sampling instants

Recall that

$$R^*(s) = R(z)\Big|_{z \leftrightarrow e^{sT}}$$

Laplace transform of sampled *r*(*t*)

Proof of claim in previous slide:

$$U^*(s) = \int_{-\infty}^{+\infty} u^*(t)\, e^{-st}\, dt = \int_{-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} u(kT)\, \delta(t-kT)\, e^{-st}\, dt$$

But because

$$\int_{-\infty}^{+\infty} f(t)\, \delta(t-\lambda)\, d\lambda = f(\lambda)$$

$$U^*(s) = \sum_{k=-\infty}^{+\infty} u(kT)\, e^{-skT}$$

$$= \sum_{k=-\infty}^{+\infty} u(kT)\, z^{-k} \quad \text{with} \quad z \triangleq e^{sT}$$

Laplace Transform of Sampled Signal

$$r^*(t) = r(t)\sum_{k=-\infty}^{\infty}\delta(t-kT)$$

$$R(s) = L[r(t)] = \int_{-\infty}^{\infty} r(t)e^{-st}dt$$

$$R^*(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} R(s - jk\,{}^{2\pi}\!/_{T})$$

R*(s)* is infinite sum of shifted copies of *R(s)

periodic with period $j2\pi/T$,

We can prove it by Fourier series expansion

Combine samplers with continuous time system:

$$U(s) = G(s)E^*(s) \qquad U^*(s) = (G(s)E^*(s))^*$$

$$U^*(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G(s - jk\,{}^{2\pi}\!/_{T})E^*(s - jk\,{}^{2\pi}\!/_{T})$$

$U^*(s)$ is $1/T$ times the sum of copies of $G(s)E^*(s)$, shifted by $j2k\pi/T$ for all integers k. But $E^*(s)$ is already periodic with period $j2\pi/T$, so $E^*(s - j2k\pi/T) = E^*(s)$ for any integer k.

Thus

$$U^*(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G(s - jk\,{}^{2\pi}\!/_{T})E^*(s) = G^*(s)E^*(s)$$

In general, for $U(s) = E(s)G(s)$ we have $U^*(s) \neq E^*(s)G^*(s)$

Instead, we have $U^*(s) = (E(s)G(s))^*$

For any **P(s) that is periodic in s with period $j2\pi/T$**

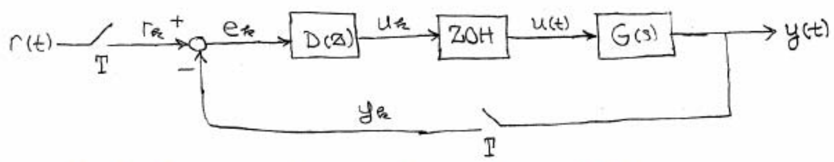
$$U^*(s) = (G(s)P(s))^* = (P(s)G(s))^*$$

we have $U^*(s) = G^*(s)P(s)$

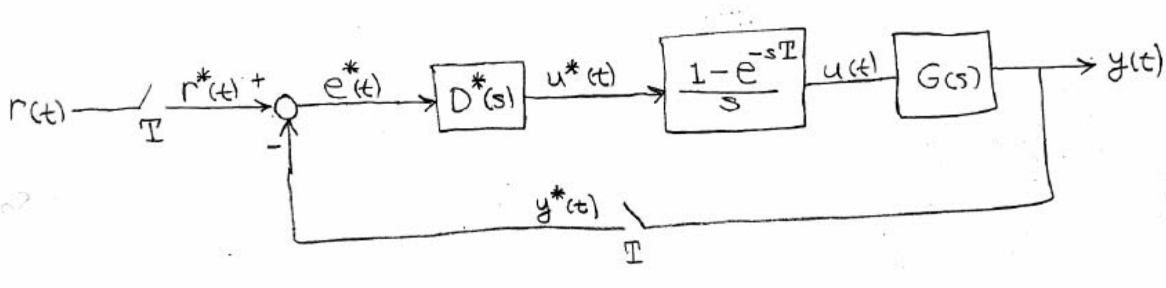
[= $G^*(s)P^*(s)$ since $P^*(s) = P(s)$]

In other words, we can factor periodic terms out of a (…)* term

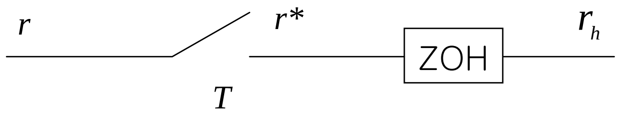
Block Diagrams and * Transform



D(z) is a digital controller—given by a difference equation. Here some things are in continuous time and some are in discrete time. **Now represent all pieces into continuous time.**



Zero Order Holder



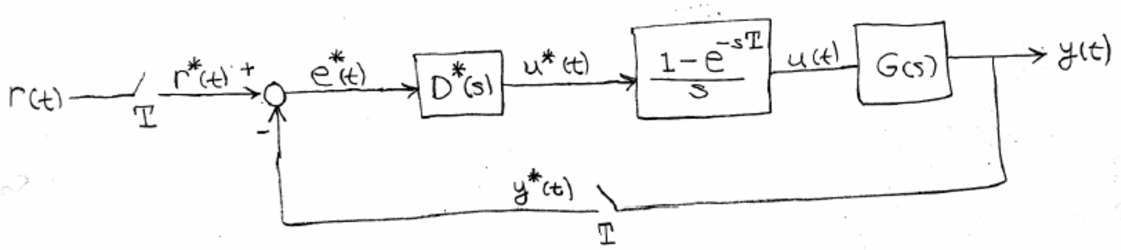
$$r_h(t) = r(0)(1(t) - 1(t-T)) + r(T)(1(t-T) - 1(t-2T)) + \cdots$$

$$= \sum_{k=0}^{\infty} x(kT)[1(t-kT) - 1(t-(k+1)T)]$$

$$L\{1(t-kT)\} = \frac{e^{-kTs}}{s}$$

$$L\{r_h(t)\} = R_h(s) = \sum_{k=0}^{\infty} r(kT) \frac{e^{-kTs} - e^{-(k+1)Ts}}{s}$$

$$= \frac{1-e^{-Ts}}{s} \sum_{k=0}^{\infty} r(kT) e^{-kTs}$$



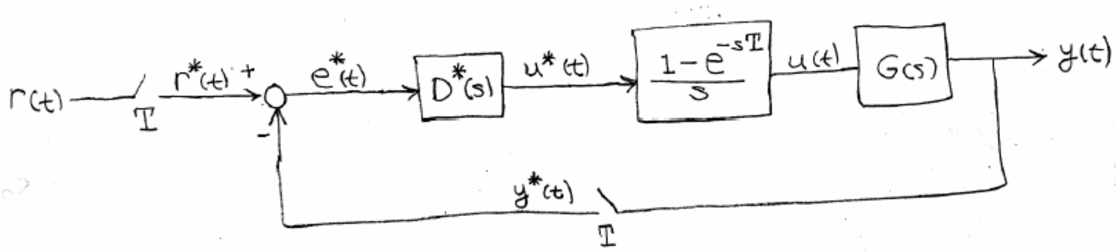
From the summation in the diagram

$$E^*(s) = R^*(s) - Y^*(s)$$

$$Y^*(s) = [G(s)\left(\frac{1-e^{-sT}}{s}\right) D^*(s) E^*(s)]^*$$

$$= \left[\frac{G(s)}{s}(1-e^{-sT}) D^*(s) E^*(s)\right]^*$$

Periodic function in s
with period $j2\pi/T$



$$E^*(s) = R^*(s) - Y^*(s)$$

$$Y^*(s) = \left[G(s)\left(\frac{1-e^{-sT}}{s}\right)D^*(s)E^*(s)\right]^*$$

$$= \left[\frac{G(s)}{s}(1-e^{-sT})D^*(s)E^*(s)\right]^*$$

$$= \left(\frac{G(s)}{s}\right)^*(1-e^{-sT})D^*(s)E^*(s)$$

$$E^*(s) = R^*(s) - Y^*(s)$$

$$Y^*(s) = \left(\frac{G(s)}{s}\right)^*(1-e^{-sT})D^*(s)E^*(s)$$

Substituting *E*(s)* in this expression, we get an expression relating input *R*(s)* to output *Y*(s)*

$$Y^*(s) = \left(\frac{G(s)}{s}\right)^*(1-e^{-sT})D^*(s)[R^*(s) - Y^*(s)]$$

can be written as

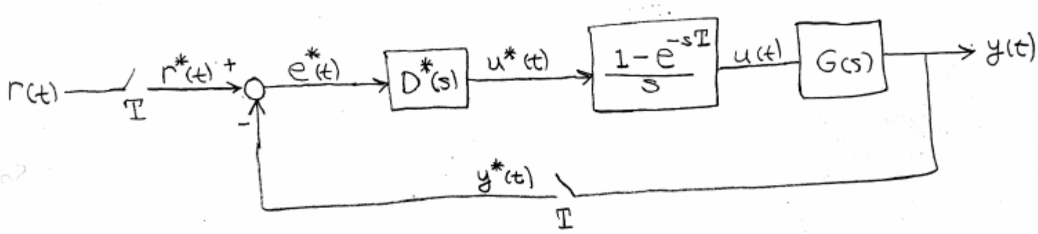
$$Y^*(s) = \left(\frac{H^*(s)}{1+H^*(s)}\right)R^*(s)$$

CL transfer function

with

$$H^*(s) \equiv \left(\frac{G(s)}{s}\right)^*(1-e^{-sT})D^*(s)$$

Also from the diagram we can get a transfer function from input $R^*(s)$ to output $Y(s)$ (not $Y^*(s)$)



$$Y(s) = \left(\frac{G(s)}{s}\right)(1-e^{-sT})D^*(s)[R^*(s) - Y^*(s)]$$

Compare this with

$$Y^*(s) = \left(\frac{G(s)}{s}\right)^*(1-e^{-sT})D^*(s)[R^*(s) - Y^*(s)]$$

$$Y(s) = \left(\frac{G(s)}{s}\right)(1-e^{-sT})D^*(s)[R^*(s) - Y^*(s)]$$

Substitute for $Y^*(s)$ in the above:

$$Y(s) = \left(\frac{G(s)}{s}\right)(1-e^{-sT})D^*(s)[R^*(s) - \frac{H(s)^*}{1+H(s)^*}R^*(s)]$$

$$Y(s) = \left(\frac{G(s)}{s}\right)(1-e^{-sT})D^*(s)[R^*(s) - \frac{H(s)^*}{1+H(s)^*}R^*(s)]$$

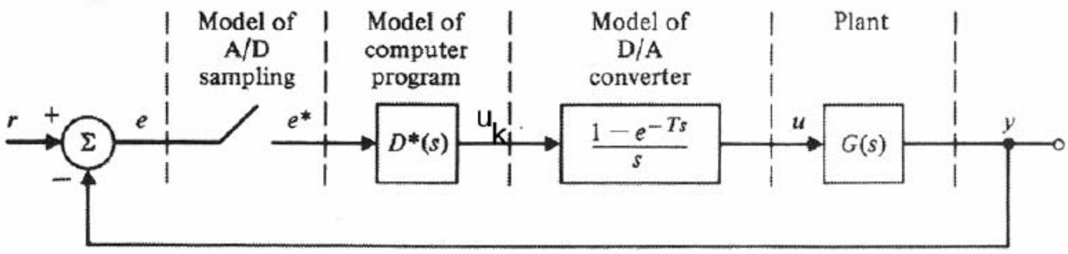
$$= \left(\frac{G(s)}{s}\right)(1-e^{-sT})D^*(s)[\frac{R^*(s)}{1+H(s)^*}]$$

$$Y(s) = \left(\frac{G(s)}{s}\right)(1 - e^{-sT})D^*(s)[\frac{R^*(s)}{1 + H(s)^*}]$$

We can use this, by inverse Laplace Transform, to get *y(t)*

This transfer function represents the frequency response of a system whose shape we can alter by choice of digital controller *D(z)* [or *D*(s)*]

Examples



We want to find *Y* and *Y**. We have, from before

$$Y^*(s) = \left(\frac{H^*(s)}{1 + H^*(s)}\right)R^*(s)$$

$$H^*(s) = \left(\frac{G(s)}{s}\right)^*(1 - e^{-sT})D^*(s)$$

$$Y(s) = \left(\frac{G(s)}{s}\right)(1 - e^{-sT})D^*(s)[\frac{R^*(s)}{1 + H(s)^*}]$$

Let $G(s) = \frac{a}{s+a}$

That is, we have a given system *G(s)* that we want to control

with digital controller given by difference equation

$$u(kT) = u(kT - T) + K_o e(kT)$$

$$u[k] = u[k-1] + K_o e[k]$$

To calculate *H** we first obtain *D(z)* from this difference equation

$$D(z) = \frac{U(z)}{E(z)} = \frac{K_o}{1 - z^{-1}} = \frac{K_o z}{z - 1}$$

Making the z ~ e^{ST} substitution

$$D^*(s) = \frac{K_o e^{sT}}{e^{sT} - 1}$$

Next, we need to find the star transform for the combined plant and zero order hold

$$(1 - e^{-Ts})(G(s)/s)^* = (1 - e^{-Ts})\left(\frac{a}{s(s+a)}\right)^* = (1 - e^{-Ts})\left(\frac{1}{s} - \frac{1}{(s+a)}\right)^*$$

The complication here is that the stuff that we partial fractioned is sampled (this is the meaning of the *) – that is, the inverse Laplace transform is sampled in the time domain

If we use our tables for the corresponding z transforms of the sampled signals, then we can convert them into s domain expressions (of the sampled signals)

Recall that $F^*(s) = F(z)\Big|_{z \sim e^{sT}}$

Then using this and tables

$$\frac{1}{s} \sim \frac{z}{z-1} \Rightarrow \frac{1}{s} \sim \frac{e^{sT}}{e^{sT}-1} = \frac{1}{1-e^{-sT}}$$

$$\frac{1}{s+a} \sim \frac{z}{z-e^{-aT}} \Rightarrow \frac{1}{s+a} \sim \frac{e^{sT}}{e^{sT}-e^{-aT}} = \frac{1}{1-e^{-sT}e^{-aT}}$$

So we then get

$$(1-e^{-Ts})(G(s)/s)^* = (1-e^{-Ts})\left(\frac{1}{1-e^{-Ts}} - \frac{1}{1-e^{-aT}e^{-Ts}}\right)$$

Choose T such that ***e^{-aT} = ½*** (for convenience), results in

$$(1-e^{-Ts})(G(s)/s)^* = \left(\frac{(1/2)e^{-Ts}}{1-(1/2)e^{-Ts}}\right) = \frac{(1/2)}{e^{Ts}-(1/2)}$$

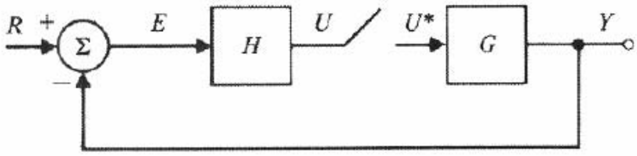
Combining this with the expression we got for *D*(s)* yields

$$H^*(s) = \frac{K_o}{2}\frac{e^{sT}}{(e^{sT}-1)(e^{sT}-1/2)}$$

Thus

$$Y(s) = R^*\frac{D^*}{1+H^*}\frac{(1-e^{-sT})}{s}G(s)$$

Example 2:



Find Y and Y^*

Step 1: write down relationships from blocks

$$E = R - Y$$
$$U = HF$$
$$Y = U^* G$$

Step 2: Take *'s of all

$$E^* = R^* - Y^*$$
$$U^* = (HE)^*$$
$$Y^* = U^* G^*$$

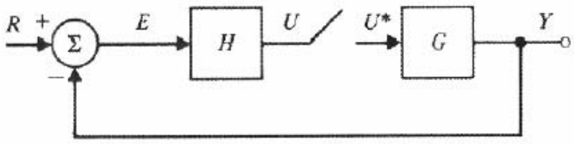
Step 3: Solve for output of sampler:

$$U^* = (H(R - Y))^*$$

$$U^* = (HR)^* - (HU^* G)^*$$
$$= (HR)^* - U^*(HG)^*$$

(since U^* is periodic)

Note: factoring out U^* may not work if H and G are matrices

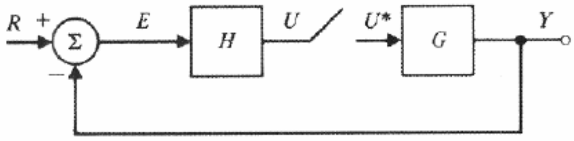


$$U^* = (HR)^* - (HU^* G)^* \quad = (HR)^* - U^*(HG)^*$$

Solving this, we have

$$U^* = \frac{(HR)^*}{1 + (HG)^*}$$

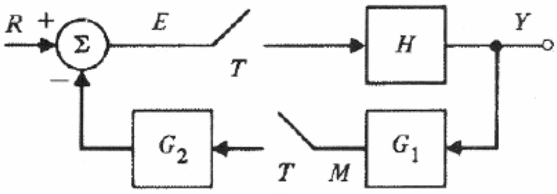
$$U^* = \frac{(HR)^*}{1 + (HG)^*}$$



And since $Y^* = U^*G^*$

$$Y^* = \frac{(HR)^*}{1 + (HG)^*} G^*$$

Another example



$$E(s) = R - M^* G_2$$
$$M(s) = E^* H G_1$$

E and M selected as the independent variables—since they are sampled

$$E^* = R^* - M^* G_2^*$$
$$M^* = E^* (HG_1)^*$$

and

$$E^* = R^* - E^*(HG_1)^* G_2^* = \frac{R^*}{1 + (HG_1)^* G_2^*}$$

Hence

$$Y = E^* H = \frac{R^* H}{1 + (HG_1)^* G_2^*}$$

and

$$Y^* = \frac{R^* H^*}{1 + (HG_1)^* G_2^*}$$